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Scientific Computing: One Part of the Revolution

ROBERT M. CORLESS† AND DAVID J. JEFFERY†

Department of Applied Mathematics, University of Western Ontario
London, Ontario, Canada N6A 5B7

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This paper discusses the changes in the curriculum of first-year engineering mathematics at the University of Western Ontario that have arisen as a result of the introduction of computer algebra technology.

1. Introduction

Many studies of technology in mathematics education have discussed *how* mathematical topics should be taught using technology. There has been less discussion of *what* mathematical topics should be taught. This paper is intended to open such a discussion by describing what has been done at the University of Western Ontario for first-year engineers in the context of the current technologically driven revolution in mathematics and mathematics education.

We are in the middle of not just one but several “revolutions” in mathematics education, in fact, and have been for perhaps 20 years. However, until now, the effects of these revolutions have been confined mostly to upper-year courses, often those courses specialized for client disciplines such as engineering or computer science. The principal effect of these revolutions on the curriculum has been the introduction of new courses, such as courses on numerical linear algebra, or optimization techniques, or finite elements for heat transfer or computational fluid dynamics. Until now, the established courses have tended not to be affected, especially at the lower levels, and in particular calculus and linear algebra have not changed much since they were first introduced.

This is no longer the case, and even the first courses in calculus and linear algebra should now be revised in the light of new technologies. Some discussion along these lines can be found in Karian (1992), and in the volume which contains (Corless *et al.*, 1993) [see especially Lopez (1993)], but for the most part people concentrate on how the *teaching* should change, and not how the *content* should change. Since calculus and linear algebra are the entrance courses for so many different disciplines, changes should not be introduced lightly. It is for this reason that we present this discussion of our experiences here at Western.

We restrict ourselves in this paper to discussing the effect of using Hewlett-Packard

† E-mail: {rcorless, djj}@uwo.ca

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HP48 series calculators on the curriculum of first-year engineering mathematics courses here at Western. We used calculators, instead of more highly powered computer algebra systems (CAS), for logistic and administrative reasons. In 1988, when the program was initiated, the calculator cost roughly the same as three textbooks. The calculator cost has kept constant or even declined over the years, and is in any event comparable to the cost of software packages for microcomputers. The main logistic reason for using calculators is the fact that the students (approximately 300) are able to take them into exams. The main administrative reason for computers not being used is that if the students were forced to buy microcomputers, then that purchase price would have been removed from the Engineering department budget for laboratory equipment. Even without these logistic and administrative reasons, the HP48 series calculator is still not a bad choice, because it provides a small but very highly integrated environment for scientific computing, with a high-level programming language derived from FORTH and LISP, acceptable graphics, powerful numerics, and acceptable symbolic capabilities.

It is by no means clear that remaining standardized on this series of calculators is the best thing to do. However, most of the items we discuss in this paper, about curriculum change as a result of use of technology, apply equally well or perhaps even better to more highly developed CAS.

Before we launch into a study of how the curriculum of our first-year engineering mathematics changed as a result of technology, some discussion of why we would want to change the curriculum seems reasonable. The first reason is that if we choose to use technology, even just to help teach standard mathematics, some change is forced on us. This is because some activities become irrelevant in the light of technology, and others become somewhat counterproductive (for example, teaching students algorithms that do not scale up to real problems, such as Cramer's Rule).

The second reason is that there are opportunities to improve the curriculum, trimming away redundant material and replacing it with fresher, more useful and central material. The principle that drives this is that the curriculum has always been chosen with due regard for what it is practical to get the students to do. In the past, some desirable topics were omitted or downplayed because they were impractical for hand calculation for the majority of students. These themes recur in the examples that follow.

As a final reason, we are technological skeptics. We feel that the potential for harm in the misuse of technology is very great. We also feel that the proper response to this is to teach students the best use of technology. We hope to teach them among other things that technology can be fallible, and machine solutions must be checked because it is not the machine that is responsible for an error, it is the person running the machine.

1.1. AN ALTERNATIVE: BAN TECHNOLOGY

It is perfectly viable to teach mathematics, even to engineering students, without any technology beyond pencil, paper, and chalk. The Luddite argument, that technology is inherently bad, of course does not hold water; but the converse argument, that technology is inherently good, is similarly invalid. Furthermore, there are arguments against technology that are neither Luddite nor based on the fact that using it in class increases the amount of work the instructor has to do. The pencil-and-paper methods, after all, worked for us—why can't they work for our students? This is not a trivial issue, and it's not obvious that technology will make things better.

We believe, however, that once the students leave the classroom they will be living in a technological world, and some attempt to get students ready for this is the academically responsible thing to do. We remark in passing that we are supported in this by the overwhelming majority of students. The rest of this paper describes what happens once this decision is taken.

1.2. DATA COLLECTION AND MEASUREMENT

This present paper is a synopsis of our experience, and we do not report any measurements of student performance or opinion here. Some measurements were taken, and reported in Corless *et al.* (1990, 1993), but these were small-scale measurements and based largely on self-report by the students. No truly scientific study, based on actual performance as opposed to student opinion, of the effects of the curriculum changes discussed in this present paper has been done. Indeed, our project is best viewed as exploratory, and preliminary to a possible larger-scale discussion and study. Objective measurements of student performance in subsequent courses and their careers would be very useful.

2. Case Studies

We give short discussions of examples of how technology has changed the mathematics we have taught to our engineering students. These are typical of the changes that have already occurred.

2.1. OBVIOUS CHANGES

When a calculator can graph a function for you, why should you learn techniques for sketching curves? Sketching curves was formerly an important topic in first-year calculus. Now it's 'obviously' a waste of time. One might ask, what should we do instead?

In fact, getting good qualitative information out of a technically accurate but possibly misleading computer or calculator plot can sometimes be difficult (for a discussion of this in a different context see Tuska (1993)). Consider the graph of the polynomial

$$p(x) = \prod_{i=1}^{7}(x - i/8)\,,$$

which has seven equally spaced zeros in $(0, 1)$. Suppose we plot this on $0 \leq x \leq 1$. Unless our y-axis scale is chosen very carefully, a calculator plot will not show the extrema and the curve will look flat. So a certain amount of time has to be spent teaching the students much the same material as before, about local maxima and minima and their relationship to the derivative, just to teach students how to use the graphing capability of the calculator properly.

It is true, however, that this can be done in less time than was previously devoted to this portion of the course, and the creative use of "zooming" features substantially increases the student's understanding. "Using the zoom button" is a new topic, if you like, though it's hardly advanced mathematics.

We note in passing that *Mathematica* has a very good graphics package, which is able to scale some difficult plots automatically (including the above polynomial example) so as to show interesting features. This is impressive, though we don't know the facility or the system well enough to comment on what it is doing or how robust it is.

2.2. LIMITS

Limits are fundamental to the study of calculus. If limits are to be done as analysis and not as algebra†, then some numerical evaluation of limits becomes natural in a technological environment. Limits will always push on the boundaries of what can be done with floating-point arithmetic, and if the instructor chooses to use numerical evaluation of limits as an expository tool, then some explanation of floating-point arithmetic becomes necessary.

Consider the standard example of the evaluation of

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\,. \tag{2.1}$$

This arises in our sequence when we compute the derivative of $\log_a(x)$ from first principles. If you ask students to *guess* the limit in equation (2.1) before computation, a significant portion of the students guess that the limit is 1 because they can see that $1 + 1/n$ is going to 1, and they argue that 1 raised to any power is 1. If you then ask them to do some computation, they find that for $n = 10^5$ and even $n = 10^{11}$ on the calculator, a number close to $e = 2.718\ldots$ is generated. Some students will inevitably want to try larger n. If numbers larger than $n = 10^{11}$ are used, the calculator produces 1, apparently agreeing with their incorrect guess.

This seeming agreement is hard to dislodge; students even tend to disbelieve a mathematical proof after this process (Dick, 1991). So our approach is slightly different: we explicitly control the numbers n going in, showing the close agreement to e; then we outline the proof that the limit really is e, and point to Niven (1981) for full details for interested students, and *only then* do we show the calculations for larger n. At that point we *explain* the numerical difficulty, namely that $1 + 1/n$ rounds to 1 for n large enough. This explanation is not part of the standard calculus curriculum, but it is necessary to do this to head enthusiastic students off from plugging larger n into the formula and then getting mixed up. Incidentally we can also use this example to explain the existence of the `LNP1` function on the HP calculator, which otherwise puzzles many students (`LNP1` accurately computes $\ln(1+x)$ for x near 0, and by writing $(1+1/n)^n = \exp(n\ln(1+1/n))$ we can accurately evaluate the limit on the calculator, using the continuity of the exponential function).

This change in the curriculum seems minor, but we have in fact had to confront a limitation of scientific computing, and do so explicitly. This difficulty may ambush any limit calculation because of the nature of such calculations. We note this difficulty does not go away with arbitrary-precision arithmetics, because the student will still have to understand the relationship between the number of digits used and the accuracy of the final result.

2.3. INTEGRALS

The definition of a definite integral as the intuitively obvious area under a graph, which can be bounded above and below by finite Riemann sums, is purely concrete and

† In a first year course one can omit all analysis by asking students simply to believe that some functions are continuous, to believe that the sum, product, and composition of continuous functions are continuous, and that division introduces discontinuity only with a zero denominator. After that, limits become algebra because $\lim_{x\to a} f(x) = f(\lim_{x\to a} x)$ for continuous f.

visual. The students understand this *much* better than they understand the standard "antidifference" techniques for finding a formula that one must then take a limit of. For example, consider computing $\int_0^1 \exp(-x)\,dx$ by both methods. In the antidifference approach, we first write down the general Riemann sum

$$S_n = \sum_{k=0}^{n-1} e^{-k/n} \cdot \frac{1}{n}$$

(this is already difficult for many students) and notice that $\exp(k/n) = \exp(1/n)^k$. We see that if we put $z = \exp(-1/n) < 1$ then

$$S_n = \frac{1}{n} \sum_{k=0}^{n-1} z^k \,,$$

which is a geometric series, with exact sum

$$S_n = \frac{1}{n} \cdot \frac{1 - z^n}{1 - z} \,.$$

After separately establishing that the limit as $n \to \infty$ of $n(1 - \exp(-1/n))$ is 1, we establish the value of the definite integral to be $1 - \exp(-1) = 0.632\,120\,558\,8\ldots$.

Now we consider the concrete, finite sum approach: take a left-hand Riemann sum with (say) 100 panels, and a right-hand Riemann sum with the same number of panels. Since the function is monotonically decreasing, the left-hand Riemann sums will provide an upper bound, and the right-hand sums provide a lower bound (we insist the students draw pictures, too, which help them to keep left/right and lower/upper separate). We get that the true value of the definite integral is between $0.628\,965 < A < 0.635\,29$; using instead the midpoint and trapezoidal rules (which provide lower and upper bounds respectively because the function is convex), we get $0.632\,117\,9 < A < 0.632\,125\,83$ using the same number of panels. This gives us four decimal place accuracy, since rounding errors play no role here.

The symbolic approach is preferred for existence questions, and also in the absence of calculators or computers, but the computational approach is *by far* the simplest conceptually. It is also of far greater generality: consider $\int_0^1 \exp(-x^3)\,dx$, for example.

It has often happened that students who are too weak to formulate the abstract definition of area under a graph as the limit of a general sum, can nonetheless, even under exam conditions, deliberately compute upper and lower bounds on that same area to four or more decimal places, using only finite sums. The difference is in the level of abstraction. Our point is that if we shift the teaching away from that abstraction, temporarily, we can prepare the student for a later, more analytical, course. In the meantime the student gains practice in *formulating* integrals from first principles—which is often what is most needed in applied problems anyway—and confidence that an integral is an answer, not a question.

Efficiency questions lead very naturally into a discussion of better numerical methods, culminating in Romberg integration and the algorithm behind the Hewlett-Packard numerical integration key (Kahan, 1980). Thus a topic which is usually treated separately, some years after the first calculus course, has become part of the first course. We should comment that students often actually want to know the details of the algorithm, because they are genuinely curious as to how the calculator can integrate.

Students also ask the question "why do I need symbolic methods, when the numerical method is so good?" This question is hard to answer convincingly: there *are* answers, but you have to work hard to get them across. It is true that the set of elementary functions whose integrals can be found by the typical engineer is much smaller than the set of definite integrals that can be evaluated numerically. We find the justifications that are accepted include the fact that numerical methods have trouble with improper integrals, symbolic methods can improve the accuracy and efficiency of integration, and that integrals can be considered as answers rather than as questions.

Another justification is that numerical integration is provably impossible (Kahan, 1980). We use examples based on Kahan's proof to show the students this, and to encourage skepticism. See Corless (1993) for details.

Finally, we explicitly included space and marks on exams for checking symbolic answers. We contend that not enough time is spent in the standard calculus course on teaching students to check their answers. The standard lecture format does not encourage the checking of answers, and neither do most textbooks.

2.4. PARTIAL FRACTIONS

The first analytic method of integration that we teach is that of partial fractions. For this we use only the power rule

$$\int x^n \, dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{if } n \neq -1 \\ \ln(x) + C & \text{otherwise} \end{cases}$$

and complex numbers. One of the principal advantages of the HP scientific computing environment is the "seamless" use of complex numbers. Thus we have

$$\int_1^3 \frac{2\,dx}{1+x^2} = \int_1^3 \left(\frac{i}{x+i} - \frac{i}{x-i} \right) dx = \big[i\ln(x+i) - i\ln(x-i) \big]_1^3$$

instead of the usual arctangent formula, and the calculator evaluates this very nicely (the imaginary parts cancel, as they should).

There is initial resistance to this use of complex numbers, partly because many students know some calculus on entering the course and they know how to use the arctangent formula. Requiring them to do a few examples with both methods rapidly convinces them that the complex formulation is simpler, particularly in the case of repeated roots. They are told that they can use whichever method they prefer on exams, and the overwhelming majority choose to use complex numbers. This method really is simpler, because it involves fewer algebraic manipulations. As an example, consider

$$\int \frac{2 + 2x + x^2 + 2x^3}{(x^2+2)^2} \, dx$$

which can be expressed either in the form of a rational function and a sum of complex logs or in the form of a rational function, a sum of real logs and arctangents. The second form requires another integration rule to be used, that for arctangents, where the complex form requires only the power rule.

The curriculum change here is not one that was forced on us, but one that we chose to take advantage of. We believe that the material here is simpler, easier to learn, easier to use, and better preparation for later use of complex variables in (say) control theory or Fourier series.

2.5. THE METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is a technique hallowed by tradition for the solution of linear inhomogeneous ordinary differential equations. It is much more rarely seen in a first-year calculus course for the solution of the simplest such differential equation, namely $y'(x) = f(x)$. In fact, we know of only one book, the iconoclastic and excellent book (Hamming, 1985), which is truly different from the "standard" calculus book, that presents the method of undetermined coefficients for integration.

Teaching the method of undetermined coefficients in a first-year class allows us to prepare the students to meet the method later, to see the fundamental unity of all integration heuristics (a guess is necessary at some point no matter what the method), and to introduce the error function, their first "non-elementary" function. Finally, it prepares the ground for teaching the Risch integration algorithm in a later course, and at the very least allows us to discuss the existence of this algorithm.

A typical example of the method of undetermined coefficients is to find an expression for $\int(x^2 + 3x + 2)e^x\,dx$, first by explicitly guessing that the answer will be of the form $F(x) = (ax^2 + bx + c)\exp(x) + k$, differentiating both sides and equating coefficients to identify a, b, and c. The students pick this up rather quickly, though many do not like the idea of guessing the form of the answer.

Later we use a more sophisticated form of the guess, which in this case would be $F(x) = p(x)\exp(x) + k$ where $p(x)$ is an undetermined polynomial. This leads directly to the Risch differential equation for this problem, namely

$$p'(x) + p(x) = x^2 + 3x + 2\,.$$

The students learn to solve such equations over the domain of polynomials.

2.6. SUMMARY OF INTEGRATION CHANGES

For integrals, then, curriculum changes due to technology allow us to lower the level of abstraction by using brute-force evaluation of finite Riemann sums; to give the students confidence that an integral is an answer and not a question; to focus on the formulation of integrals; to encourage students to check their answers; to prepare them for the Risch integration algorithm later; and finally to expand the student's vocabulary of functions to include some non-elementary functions. These changes enrich the course. We also explicitly teach the students that technology is fallible, and give them examples where the integration methods fail. This is possible and desirable with any CAS, not just a calculator.

2.7. CONSTRAINED AND UNCONSTRAINED OPTIMIZATION

The following problem is taken from Boas (1966, p. 184). Variants of it appear in many textbooks.

> Find the largest box [with faces parallel to the coordinate axes] that can be inscribed in
>
> $$g(x,y,z) = \frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{25}z^2 = 1\,.$$

Using a Lagrange multiplier, one reduces the problem for the coordinates (x, y, z) of one corner of the box to the simultaneous nonlinear equations

$$8yz + \frac{1}{2}x\lambda = 0\,, \tag{2.2}$$

$$8xz + \frac{2}{9}y\lambda = 0\,, \tag{2.3}$$

$$8xy + \frac{2}{25}z\lambda = 0\,, \tag{2.4}$$

$$\frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{25}z^3 - 1 = 0\,. \tag{2.5}$$

Few mathematicians would feel comfortable with leaving the problem at this point, and most would want to see the equations solved to obtain $x = 2/\sqrt{3}$, $y = \sqrt{3}$, $z = 5/\sqrt{3}$ and the volume $80/\sqrt{3}$. Since students are not taught general methods for solving multivariate polynomial equations in the first year, this problem is suitable as an exercise or exam question only if some manipulation renders the equations easily solvable. In this case, the trick is to multiply equation (2.2) by x, equation (2.3) by y, equation (2.4) by z, and add the results. Then using the constraint equation we can identify $\lambda = -12xyz$. Any student who obtains equations (2.2)–(2.5) but fails to solve them will naturally wish for as much credit as possible, on the grounds that the method of Lagrange multipliers stopped at the equations and their solution was a separate issue.

Instructors have long recognized this separation and have chosen to give example optimization problems which led to systems of equations which were merely linear, or where a similar one-off trick could be used. This is less satisfactory to the modern instructor, in view of the potential for using technology to handle the separate problem.

We chose to teach our students about multivariate Newton's method, as a follow-up to a more usual detailed study of Newton's method in one dimension. We gave them a calculator program implementation of Newton's iteration—that is, the calculator program would take as input an estimate of the answer and return the next iterate in the Newton process—and an initial guess to use. If the student had formulated the problem correctly, they could get the right answer, if they also understood they had to continue iterating until the answer converged.

In other contexts, other choices might be better. If you are using a fully powered CAS in the classroom, it might be possible to introduce resultants or even Gröbner bases at this stage, though perhaps a simple "solve" command would be acceptable.

We remark that having implemented some form of "solve" command, in our case by multivariate Newton's method, the problem of solving unconstrained problems becomes easy. And then the true beauty of Lagrange's idea of multipliers comes out, because it converts constrained extrema to unconstrained ones. Note that this approach follows the "White Box/Black Box" approach of Buchberger (1990), in that at this point the idea of solving a system of equations is understood; but it also violates the principle to some extent because most students are really not clear on the details of the multivariate Newton iteration. Since the students are able to understand the purpose and behaviour of the multivariate Newton's iteration from their experience with and detailed knowledge of the one-dimensional case, we feel this violation of the principle is not serious, and otherwise in the course we follow the principle fairly rigorously.

In the unconstrained case we also had access to the student's knowledge of quadratic forms from their concurrent linear algebra course. By testing if a certain symmetric quadratic form is positive definite or not, again using a calculator program that we supply, the students can decide if the extremum is a local maximum, minimum, or saddle. These techniques are more general than the techniques usually taught at this level, and we feel the students are better served by seeing them instead of seeing only problems which have to be solved by "tricks".

2.8. series algebra, or Newtonian calculus

The standard introduction to the beautiful subject of infinite series is to treat it as a game: is this series "Nardac" or "Clumglum"†? (Giving nonsense names to "convergent" and "divergent" mimics how the students see it.) The students are taught some mechanical manipulations such as the ratio test and some algebraic rules; if the series terms behave this way, the series is "Nardac", and otherwise it is "Clumglum", and sometimes the test itself fails. Playing the game well means getting a good mark on the exam.

We agree that this is an interesting game, but believe that it is hardly central to mathematics. Yet an inordinate amount of time is spent in the standard course on this game, at the expense of such topics as formulation of integrals, differential equations, or even computation of series. Few students learn any of the very interesting and general methods for discovering the terms in a power series, and are basically just taught to differentiate. We feel that the students are better served learning how to add, subtract, multiply, divide, compose, and revert *finite* series; how to estimate the error in a *finite* series; and how to think of series as answers instead of as questions.

This would better prepare a student for the use of series as generating functions, for example, than the standard approach does. Generating functions are very important for modern mathematics (Graham *et al.*, 1992). Computation of the terms in a series by hand requires either cleverness or stamina or both, and this approach is less appealing without technology. But consider the following example to see what is possible with technology.

Suppose we are given a parametric representation for a curve. For example, the cycloid has the representation (Spiegel, 1968, p. 40)

$$x = \varphi - \sin\varphi\,,$$
$$y = 1 - \cos\varphi\,.$$

Let us find a polynomial approximation for y in terms of x that is valid near the point $x = \pi$, which occurs when $\varphi = \pi$. This is easy with series reversion. We first expand $x = \pi - \sin\varphi = \pi + 2\Delta\varphi - \frac{1}{6}\Delta\varphi^3 + \cdots$ and revert this series to find one for $\Delta\varphi$ in terms of $\Delta x = x - \pi$, namely

$$\Delta\varphi = \Delta x/2 + \Delta x^3/96 + \Delta x^5/1920 + \cdots\,.$$

From $\varphi = \pi + \Delta\varphi$, $y = 1 - \cos\varphi$, and this series, we are done: the answer is

$$y = 2 - \Delta x^2/8 - \Delta x^4/384 + \cdots\,.$$

We emphasize that this technique has actually been taught to engineering students, and the majority of them were able to solve problems of this kind under exam conditions. Other series techniques we taught them included how to compute the series for $\arctan(x)$ from the series for $1/(1+x^2)$, and the like. In an appendix to our course notes (Corless *et al.*, 1991), we gave a complete description of series algebra up to and including the

† A Clumglum is inferior to a Nardac by one degree (Swift, 1726, p. 102).

J. C. P. Miller formula for finding the coefficients of a series raised to a power. At least some students understood this material (indeed, one of them, Janik Joire, wrote some of the programs for the relevant section of the course notes).

It seems to us that teaching students to use series to solve problems, to approximate functions, to learn about functions themselves, is better than teaching them a game of convergence versus divergence. It seems to us that giving them the basics of series algebra prepares them better for generating functions, for the z-transform, and even for perturbation methods. As preparation for algebra, (Laurent) series give a nice example of a field (should any of our engineering students take an interest in abstract algebra). It also seems to us that the idea of convergence of infinite series belongs in a later analysis course, not in a first year calculus course.

The idea of *error in an approximation*, we hasten to add, is of central importance in a calculus course. But convergence is not needed for this notion, and indeed we feel it can sometimes distract from the main issue. For a discussion of how numerical errors influence the course content, see the remarks on conditioning in Corless (1993).

Finally, this style of calculus brings us much closer to the approach taken by Newton himself. In some ways, we are now only beginning to appreciate Newton's approach (Arnol'd, 1990), and we feel it is time to consider teaching the calculus from more of a Newtonian perspective. The following quote is again from Arnol'd (1990, pp. 47–48).

> On the basis of Pascal's studies and his own arguments Leibniz quite rapidly developed formal analysis in the form in which we now know it. That is, in a form specially suitable to teach analysis by people who do not understand it to people who will never understand it.

While this quote is perhaps more acerbic than just, it has some merit, and analysis by means of power series is worth a central place in the calculus curriculum.

3. Concluding Remarks

The main comment we have is that technology decompartmentalizes mathematics. You may start out thinking that you will teach only derivatives and integrals, with technology confined to supplying a helping hand, but you find yourself teaching a little about scientific computing, about complex numbers, about matrices, about the solution of nonlinear equations, about series algebra, and so on. We feel this is highly desirable, and allows greater vitality in even a first year course.

A second comment is that some successes can be invisible. We did not encounter any difficulties with the use of complex numbers. The students accepted them and were able to use them competently. This is progress, compared to our past experience.

Finally we feel that the teaching of scientific computing can begin earlier than it has heretofore done, and perhaps it should start even earlier than the first calculus course.

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